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Note

Uniform Subexponential Growth of Orthogonal Polynomials

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We show that if orthonormal polynomials p_n have asymptotically periodic recurrence coefficients, then they have uniform subexponential growth on the support of orthogonalizing measure. This is an alternative proof of results of P. Nevai, V. Totik, and J. Zhang (*J. Approx. Theory* **67**, 1991, 215–234), D. S. Lubinsky and P. Nevai (*J. London Math. Soc.* **46**, 1992, 149–160), and J. Zhang (*Linear Algebra Appl.* **186**, 1993, 97–115).

1. INTRODUCTION

Let μ be a probability measure on the real line $\mathbb{R}$ with an infinite support set and all moments finite. Let $\{p_n\}_{n=0}^{\infty}$ be a system of orthonormal polynomials obtained from the sequence of consecutive monomials $1, x, x^2, \ldots$ by the Gram–Schmidt procedure. Then the p_n obey a three-term recurrence formula of the form

$$xp_n = a_{n+1} p_{n+1} + b_n p_n + a_n p_{n-1}, \tag{1}$$

where the a_n are positive coefficients while b_n are real ones.

We study the growth of $p_n(x)$, for x in the support of the measure μ. This problem has attracted considerable attention during the last 15 years. The first result in this subject belongs to Nevai [3] and deals with the case of convergent coefficients a_n and b_n, i.e., $a_n \to a/2 > 0$ and $b_n \to b$. By Blumenthal's theorem the support of μ consists then of the interval $[b-a, b+a]$ and a countable set of points with possible accumulation

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points only in $\{b \pm a\}$. It was proved in [3, Theorem 4.1.3] that for $x \in [b-a, b+a]$

$$\lim_{n \to \infty} \frac{p_n^2(x)}{\sum_{k=0}^{n-1} p_k^2(x)} = 0 \tag{2}$$

and the convergence is almost uniform in the open interval $(b-a, b+a)$. Moreover it has been conjectured that the uniform convergence holds true in the entire closed interval $[b-a, b+a]$.

This conjecture remained unsolved until 1991 when Nevai, Totik, and Zhang [4] proved it even in a more general setting, allowing complex valued recurrence coefficients, and changing the squares to arbirary positive powers p. In the case of orthogonal polynomials they proved that the convergence in (2) is uniform on the entire support of μ, which could differ from $[b-a, b+a]$ by countably many points. Next the result was extended to the case of so-called *asymptotically periodic* recurrence coefficients by Lubinsky and Nevai [2], who proved that (2) holds almost uniformly in the interior of $\operatorname{supp} \mu$. Recently Zhang [7] showed that the convergence is also uniform in the entire $\operatorname{supp} \mu$ for asymptotically periodic coefficients.

In this paper we give an alternate proof of Zhang's result. The method is a refinement of ideas of Nevai contained in [3, Proofs of Lemma 3 and Theorem 9, p. 26]. It is rather simple, especially for the case of convergent coefficients and is based on estimates involving the Christoffel–Darboux identity.

2. Subexponential Growth

Let the p_n be orthonormal polynomials satisfying (1). Fix n and consider the polynomials $p_0, p_1, ..., p_{n-1}$. There exists a discrete measure μ_n concentrated on exactly n points such that

$$\int_{-\infty}^{\infty} p_k(x)\, p_l(x)\, d\mu_n(x) = \delta_{k,l} \qquad 0 \leqslant k, l \leqslant n-1. \tag{3}$$

According to the Gauss mechanical quadrature [5] this measure is concentrated on the zeros $x_{1n}, ..., x_{nn}$ of the polynomial p_n and has the form

$$\mu_n = \sum_{k=1}^{n} \lambda_{kn} \delta_{x_{kn}} \qquad \lambda_{kn}^{-1} = \sum_{k=0}^{n-1} p_k^2(x_{kn}). \tag{4}$$

One of the key ingredients of our proof of subexponential growth is an estimate given below.

PROPOSITION 1. *Let $f(x)$ be a real-valued continuous function with compact support in $\mathbb{R}$. Then for any $x \in \mathbb{R}$ we have*

$$\frac{p_n^2(x)}{\sum_{k=0}^{n-1} p_k^2(x)} \leqslant \frac{\max_y [(x-y) f(y)]^2}{a_n^2 \int_{-\infty}^{\infty} f^2(y)\, p_{n-1}^2(y)\, d\mu_n(y)}. \tag{5}$$

Proof. We will make use of the well known Christoffel–Darboux identity.

$$a_n\{p_n(x)\, p_{n-1}(y) - p_{n-1}(x)\, p_n(y)\} = (x-y) \sum_{k=0}^{n-1} p_k(x)\, p_k(y).$$

Fix x. Multiply both sides by $f(y)$, raise them to the square, and integrate against the measure $d\mu_n(y)$. In doing so observe that the term $p_{n-1}(x)\, p_n(y)$ vanishes as μ_n is concentrated on the zeros of $p_n(y)$. Hence we obtain

$$a_n^2 p_n^2(x) \int f^2(y)\, p_{n-1}^2(y)\, d\mu_n(y)$$
$$= \int (x-y)^2 f^2(y) \left\{ \sum_{k=0}^{n-1} p_k(x)\, p_k(y) \right\}^2 d\mu_n(y).$$

The right-hand side can be majorized by

$$\max_y [(x-y) f(y)]^2 \int \left\{ \sum_{k=0}^{n-1} p_k(x)\, p_k(y) \right\}^2 d\mu_n(y)$$
$$= \max_y [(x-y) f(y)]^2 \sum_{k=0}^{n-1} p_k^2(x).$$

The last equality follows from the orthogonality relation (3). Finally, we get

$$a_n^2 p_n^2(x) \int f^2(y)\, p_{n-1}^2(y)\, d\mu_n(y) \leqslant \max_y [(x-y) f(y)]^2 \sum_{k=0}^{n-1} p_k^2(x).$$

This shows (5). ∎

Fix $\varepsilon > 0$, and let f be the function on $\mathbb{R}$ defined by

$$f(y) = \begin{cases} 1 & \text{for } |y| \leqslant 2\varepsilon \\ 2 - (2\varepsilon)^{-1} |y| & \text{for } 2\varepsilon < |y| \leqslant 4\varepsilon \\ 0 & \text{for } |y| \geqslant 4\varepsilon. \end{cases} \tag{6}$$

We denote by f_x the translate of a function f by a real number x, i.e.,

$$f_x(y) = f(y - x).$$

Now we are in a position to deduce the following.

THEOREM 1 (Nevai, Totik, and Zhang [4]). *If* $a_n \to a/2$ *and* $b_n \to b$, *with* $a > 0$ *then*

$$\lim_{n \to \infty} \max_{x \in \sup \mu} \frac{p_n^2(x)}{\sum_{k=0}^{n-1} p_k^2(x)} = 0.$$

Proof. By considering the measure $d\mu(ax + b)$ and the polynomials $p_n(ax + b)$ we can restrict our attention to the case $a = 1$ and $b = 0$. The recurrence coefficients a_n and b_n are uniformly bounded. Hence the support of the measure μ is also bounded. Assume that $\operatorname{supp} \mu \subset [-M, M]$. By [3, Theorem 3, p. 17] the sequence of measures $d\nu_n(y) = p_{n-1}^2(y)\, d\mu_n(y)$ is weakly convergent to the probability measure $(2/\pi) \sqrt{1 - y^2}\, dy$.

The supports of the measures $d\nu_n(y)$ are also contained in $[-M, M]$. Observe that

$$\lim_{y_1 \to y_2} \max_{t \in \mathbb{R}} |f^2(y_1 - t) - f^2(y_2 - t)| = 0.$$

Hence

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_x^2(y)\, p_{n-1}^2(y)\, d\mu_n(y) = \frac{2}{\pi} \int_{-1}^{1} f_x^2(y) \sqrt{1 - y^2}\, dy, \tag{7}$$

and the convergence is uniform with respect to x from the interval $[-1 - \varepsilon, 1 + \varepsilon]$.

By the definition of f_x we have

$$\begin{aligned} \int_{-1}^{1} f_x^2(y) \sqrt{1 - y^2}\, dy &\geqslant \int_{-1}^{1} \chi_{[x - 2\varepsilon, x + 2\varepsilon]}(y) \sqrt{1 - y^2}\, dy \\ &\geqslant \int_{-1}^{1} \chi_{[x - 2\varepsilon, x + 2\varepsilon]}(y) \sqrt{1 - |y|}\, dy \\ &\geqslant \int_{1 - \varepsilon}^{1} \sqrt{1 - y}\, dy = \tfrac{2}{3} \varepsilon \sqrt{\varepsilon}. \end{aligned}$$

Now by the uniform convergence in (7) we have

$$\liminf_{n \to \infty} \inf_{x \in [-1 - \varepsilon, 1 + \varepsilon]} \int_{-\infty}^{\infty} f_x^2(y)\, p_{n-1}^2(y)\, d\mu_n(y) \geqslant \frac{4}{3\pi} \varepsilon \sqrt{\varepsilon}.$$

Observe that

$$\max_y [(x-y) f_x(y)]^2 \leqslant 4\varepsilon^2.$$

Combining the last two inequalities and (5), and taking into account that $a_n \to 1/2$ we obtain

$$\limsup_{n\to\infty} \sup_{x\in[-1-\varepsilon, 1+\varepsilon]} \frac{p_n^2(x)}{\sum_{k=0}^{n-1} p_k^2(x)} \leqslant 12\pi\sqrt{\varepsilon}. \tag{8}$$

By Blumenthal's theorem the set $\operatorname{supp}\mu\backslash[-1-\varepsilon, 1+\varepsilon]$ consists of finitely many points x for which the sequence $p_n(x)$ is square summable. Hence

$$\limsup_{n\to\infty} \sup_{x\in\operatorname{supp}\mu\backslash[-1-\varepsilon, 1+\varepsilon]} \frac{p_n^2(x)}{\sum_{k=0}^{n-1} p_k^2(x)} = 0. \tag{9}$$

Now combining (8) and (9) and letting ε tend to 0 gives the conclusion. ∎

Theorem 1 generalizes to the case of asymptotically periodic recurrence coefficients. In order to carry over the proof we will need new terminology. In what is described below we follow [6, Chap. 2] (see also [1, pp. 245–246]).

For the polynomials p_n satisfying (1) the associated polynomials $p_n^{(k)}$ of order k are given by the recurrence relation

$$xp_n^{(k)} = a_{n+k+1} p_{n+1}^{(k)} + b_{n+k} p_n^{(k)} + a_{n+k} p_{n-1}^{(k)}, \qquad n \geqslant 0$$

with initial conditions $p_0^{(k)} = 1$, $p_{-1}^{(k)} = 0$.

Fix a positive integer N. We say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are asymptotically N-periodic if for every integer j the limits

$$a_j^{(0)} = \lim_{n\to\infty} a_{Nn+j}, \qquad b_j^{(0)} = \lim_{n\to\infty} b_{Nn+j}$$

do exist. Obviously the sequences $a_j^{(0)}$ and $b_j^{(0)}$ are periodic with period N.

Let $q_n(x)$ denote the orthonormal polynomials with periodic recurrence coefficients $a_n^{(0)}$, $b_n^{(0)}$. Let

$$T(x) = \frac{1}{2}\left\{q_N(x) - \frac{a_N^{(0)}}{a_{N+1}^{(0)}} q_{N-2}^{(1)}\right\},$$

$$E = \{x \in \mathbb{R} \mid -1 \leqslant T(x) \leqslant 1\}.$$

It turns out (see [1, Lemma 2, pp. 255–256]) that E consists of N closed intervals (they can meet only at the endpoints) and between every two consecutive intervals there is exactly one zero of each of the polynomials $T'(x)$ and $q_n^{(j)}(x)$, $j \geqslant 0$ (if two intervals meet at x_0 then $T'(x)$ and $q_n^{(j)}(x)$

vanish at x_0). The support of the measure μ, with respect to which the polynomials p_n are orthonormal, is of the form

$$\operatorname{supp} \mu = E \cup E^*,$$

where E^* is a denumerable set of points of which the accumulation points are in E.

In order to carry out the proof of subexponential growth for asymptotically periodic coefficients we will need an analog of the limit relation (7). Actually the limit in (7) exists if n ranges over arithmetic progression of step N (see the proof of [3, Lemma 3.2.1, p. 16]). However, we need an explicit form of the measures that are obtained by this limit procedure.

The material in this paragraph was suggested by Walter Van Assche. Let $E_j^* = \{x \in \mathbb{R} : q_{N-1}^{(j+1)}(x) = 0\}$. By [6, Theorem 2.24] we have

$$\lim_{k \to \infty} \frac{p_{kN+j-1}(z)}{p_{kN+j}(z)} = \frac{a_{j+1}^{(0)}}{2(a_j^{(0)})^2 q_{N-1}^{(j+1)}} \left\{ q_N^{(j)} + \frac{a_j^{(0)}}{a_{j+1}^{(0)}} q_{N-2}^{(j+1)} - 2\sqrt{T^2(z)-1} \right\} \tag{10}$$

for $z \in \mathbb{C} \setminus (E \cup E^* \cup E_j^*)$, where $\sqrt{T^2(z)-1}$ is that branch of the square root for which $|T(z) + \sqrt{T^2(z)-1}| \geqslant 1$.

By [6, p. 79], the Stieltjes transforms of the probability measures $d\lambda_n(x) = p_{n-1}^2(x)\, d\mu_n(x)$ (see (3) and (4) for the definition of μ_n) are given by

$$S(\lambda_{kN+j}, z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\lambda_{kN+j}(t) = \frac{p_{kN+j-1}(z)}{p_{kN+j}(z)}.$$

Thus the sequence of the measures $\lambda_{kN+j}(t)$ is weakly convergent to a probability measure $\lambda^{(j)}$ whose Stieltjes transform is given by the right-hand side of (10). By the inverse Stieltjes transform formula it can be computed that $\lambda^{(j)}$ is of the form

$$d\lambda^{(j)}(x) = d\lambda_0^{(j)}(x) + \frac{a_{j+1}^{(0)}}{2(a_j^{(0)})^2} \frac{\sqrt{1-T^2(x)}}{\pi\, |q_{N-1}^{(j+1)}(x)|} \chi_E(x)\, dx,$$

where $\lambda_0^{(j)}$ is a discrete measure concentrated on the zeros of $q_{N-1}^{(j+1)}$, and χ_E denotes the indicator function of the set E.

We are now ready to show the following

THEOREM 2 (Zhang [7]). *Let N be a fixed positive integer. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be asymptotically N-periodic sequences with a_n, $b_n \in \mathbb{R}$, and* $\inf_n a_n > 0$. *Then*

$$\lim_{n \to \infty} \max_{x \in \operatorname{supp} \mu} \frac{p_n^2(x)}{\sum_{k=0}^{n-1} p_k^2(x)} = 0.$$

Proof. Let f be the function defined by (6). Then by the remarks preceding the statement of the theorem we have

$$\lim_{k \to \infty} f_x^2(y)\, p_{kN+j-1}^2(y)\, d\mu_{kN+j-1}(y)$$
$$= \int f_x^2(y)\, d\lambda_0^{(j)}(y) + \frac{a_{j+1}^{(0)}}{2(a_j^{(0)})^2} \int_E f_x^2(y) \frac{\sqrt{1-T^2(y)}}{|q_{N-1}^{(j+1)}(y)|}\, dy,$$

where $f_x(y) = f(y-x)$. Moreover, $\operatorname{supp} \mu = E \cup E^*$, where E^* is a denumerable set which could accumulate only in E. Thus following the lines of the proof of Theorem 1, we are done if the function $\sqrt{1-T^2(y)}/|q_{N-1}^{(j+1)}(y)|$ has zeros of order 1/2. But the zeros of this expression are the endpoints of the N intervals constituting E. If an endpoint belongs to only one of the intervals, then it is a zero of order 1/2. If an endpoint belongs to two intervals then it is zero of order 1 of the numerator. But it is also a zero of the denominator as the zeros of $q_{N-1}^{(j+1)}(y)$ lie between every two consecutive intervals of E. Hence the common endpoints of two intervals are removable singularities of $\sqrt{1-T^2(y)}/|q_{N-1}^{(j+1)}(y)|$. Now the rest of the proof of Theorem 1 applies. ∎

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